Mathematics Foundation Course A:
Mathematics: A Historical Tour of the Great Civilizations

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Mathematics is the oldest of the liberal arts, yet few are aware of its vast and subtle influences on our lives. It is a practical tool, to be sure, but also it has played a major role in shaping who we are and how we think. Historically, mathematics has helped end old regimes and modes of thought, and constructed new ones. We are creatures of our own creation.

This course, designed especially for humanities and arts students (but helpful for science and mathematics students as well), takes a grand tour through the dominant mathematical cultures: ancient Babylon and Egypt, ancient Greece, medieval Islam, pre-modern China, and Europe to today. We shall discover how mathematics shaped, and was shaped by, the people who practiced it, and how it interacts with worldviews and alters ideas. Our voyage will alter our preconceptions of what mathematics is, and how important it is to us.

TEXTS:

***Required*:** William P. Berlinghoff and Fernando Q. Gouvêa, *Math Through the Ages: A Gentle History for Teachers and Others*, Farmington, ME: Oxton House, 2002.

***Recommended/Reference (Library/Learning Commons)*:** Frank Swetz (ed.), *From Five Fingers to Infinity*. Chicago: Open Court Press, 1994.

Five Fingers has gone out of print. I have put together copies of the articles that we will be using, as well as the table of contents. A number of articles from other sources will be available as well. Two copies will be in the Learning Commons, and one on reserve at the library

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DAILY PATTERN:

We will meet twice per day for 90 minutes. The classes will consist of the following:

- Story-telling;
- Problem-solving in the style of the various cultures we will be studying;
- Short presentations through the first two weeks by students, usually in pairs (see below);
- Several videos;
- Discussions of articles.

ASSESSMENT:

• Six daily, mostly mathematical problem sets	20%
• Six daily, mostly historical assignments	20%
• Short presentation	15%
• Final project	35%
• Participation/initiative	10%

Problem sets: These should take about 3-4 hours, and build on the techniques and ideas we saw in class. Feel free to discuss the individual problems with each other, but make sure that what you turn in the next morning is work you can call your own.

Historical assignments: These will usually consist of an article to read, and a couple of questions which require a mini-essay response (1-2 paragraphs).

Short presentations: These will usually happen in the first two weeks of the course; you may work on your own or in pairs. The topics may be taken from the article "Seeking Relevance? Try the History of Mathematics", or one of the 25 episodes in our textbook.

Final project: This will be undertaken mostly at your initiative; see the large handout. About ¾ of students in the past have written a paper (about 10-15 pages), but creative suggestions are welcomed and encouraged! Students have built historical mathematical instruments, written plays, made videos, produced historical units for school classrooms, etc.

Near the end of the block, if time permits, you might have an opportunity to present your project to the class.

Learning Outcomes Grid

Learning Outcome	*Class activities*	*Assessment*
College-wide		
Critical thinking	Discussion of issues; small group problem solving	Daily mathematical and historical assignments; final project
Communication	Short presentations; all other assignments (including mathematical!)	Written assignments; short presentations
Research	Short presentation; especially the longer paper	Short presentation; especially the longer paper
Integration	Class discussions and videos emphasizing connections with cultural history	Some questions in written assignments; emphasized in final project
International perspectives	Course revolves around different cultural presentations of mathematics	Short historical assignments emphasize cultural comparisons
Math foundation		
Understanding abstraction	Mathematical cultures arise from historical, practical contexts	Mostly in mathematical assignments, final project
Recognizing types of abstraction	Central theme: tensions between arithmetic and geometric views; the rise of algebra	Mathematical assignments, short presentations, most final projects
Choosing appropriate models	Students discuss historical views on the above and their best resolution	Mathematical assignments, short presentations, most final projects
Selecting problem-solving strategies	In-class mathematical activities based on historical models	Mathematical assignments, some short presentations, most final projects
Applying/evaluating tools	Use of ruler and compass; iterative versus analytic solutions	Mathematical assignments, some final projects
Placing results in context	All problems arise from culture-specific contexts; many lead to historical conclusions	Historical assignments, some mathematical assignments, short presentation, final project

Tentative Course Outline

Day 1	• Introductions; the nature of mathematics • Mathematical autobiographies; **Video:** "Donald Duck in MathMagic Land" video
Day 2	• The origins of numbers and number systems • Number systems in ancient Egypt
Day 3	• Egyptian algebra and geometry • Babylon and its "geometry"
Day 4	• Babylonian computation and algebra • The beginnings of the "miracle": ancient Greek mathematics
Day 5	• Pythagorean mathematics • The "crisis"? Developing logical foundations in mathematics
Day 6	• Euclid's *Elements* (most of the day) • **Video:** "The Tunnel of Samos"
Day 7	• Even more Euclid • Archimedes and Apollonius **Video:** "Lost Secrets"
Day 8	• Astronomy and the birth of trigonometry • Introduction to medieval Islam; the reception of foreign knowledge • Islamic arithmetic and algebra
Day 9	• Islamic arithmetic and the emergence of algebra • Islamic algebra and geometry • Islamic trigonometry and astronomy
Day 10	• Islamic trigonometry and astronomy • Cosmic geography: al-Biruni and the dimensions of the Earth
Day 11	• **Video:** "Kubba for al-Kashi" **Activity:** Al-Kashi, π, and sin(1) • Introduction to China; linear algebra
Day 12	• Chinese geometry and computation • What did "proof" mean in China, if anything?
Day 13	• A Chinese geometrical problem • European revival in the Renaissance; the story of the cubic

Day 14	• The birth of analytic geometry • **Article: "**How Galileo Changed the Rules of Science" **Video: "**Cosmic Highway: Galileo"
Day 15	• How calculus changed the world • The crisis in calculus
Day 16	• Non-Euclidean geometry • The problem with logic: Russell's paradox, the foundations crisis, and Gödel's Theorem
Day 17	• The mathematics of infinity • **Video:** "The Proof" (Fermat's Last Theorem)
Day 18	• Student project presentations

HINT SHEET
Why are the Differences Between the Perfect Squares Equal to the Successive Odd Numbers?

HOW TO USE THE HINT SHEETS: Do *NOT* look at any of the hints now! First try to think through an explanation. If you get stuck, fold the paper so that only Hint 1 is visible. Only move on to the next hint if you're stuck.

① How can you *draw a picture of* a perfect square, like (say) 16?

①ⓐ A perfect square is just that: a square block of dots, or blocks, or whatever.

② How can you draw a picture of the next-larger perfect square (25)?

③ The difference is what we're after. Without changing the shapes of your drawings, take one shape away from the other.

④ How many dots are there in the left-over shape? What is the relation to the size of your original square?

⑤ Try it out for other size squares, such as 9 and 16.

⑥ Break your L-shaped difference into pieces… but how?

⑥ⓐ If you take away the dot at the corner of the L, the two left-over pieces are equal to the side of the original square.

CHALLENGE: Pick any quadratic equation, say, $5x^2 + 3x + 2$. Plug in $x = 1, 2, 3, \cdots$ and notice the pattern in the differences of the results. Now that you know the pattern for x^2, what is the pattern for any $ax^2 + bx + c$? Can you *explain* this?

ACTIVITY/HINT SHEET
Hipparchus and the Eccentricity of the Sun's Orbit

① Can you convert the day lengths between the equinoxes and solstices in degrees? Remember, there are 365¼ days in a year, but 360 degrees in a circle.

② Connect the center of the universe with the summer solstice, and with *both* equinoxes.

③ Can you determine the angle *a*, at point *C*, from the vertical line to the summer solstice? At the same time, determine *b*, the angle at *C* from the horizontal to one of the equinoxes.

> ③ⓐ You can get angles *a* and *b* from the angles you got before.
>
> ③ ⓑ If you can't get it, just use $a = 2.15°$ and $b = 0.99°$.

④ Notice the small triangle at the middle, for which *e* is the hypotenuse. Can you use trigonometry to figure out the lengths of the other two sides?

> ④ⓐ For the horizontal side, use the large triangle that includes that side, the point *C*, and the summer solstice.
>
> ④ ⓑ Use a sine, and remember that the radius of the orbit is 1.

⑤ Now you can figure out *e*!

⑤ⓐ Think of a famous dead Greek guy.

⑤ⓑ Remember the beanfield?

ACTIVITY SHEET

Hilbert and his Friends

Is it *always* possible to make room for Hilbert's friends, no matter how popular he is?

① Suppose that, instead of just one friend, Hilbert shows up with 10 friends. Can they all be accommodated? If so, how?

② Now, a stiffer challenge. The next night, *every* single person in the entire hotel shows up with a friend. Now can they all fit? Explain how it can be done.

③ (If time) Now suppose that the first person comes with two friends, the second person with four friends, the third person with six friends, and so on. Can you fit them in now? What are the new room numbers of the original guests?

From the above, you might think that you can fit everyone in, no matter how many people arrive. It would make sense. And, you would be wrong⋯

Sample Problem Set Questions (diagrams not included)

1. On the back of this sheet are translations of the ancient Chinese characters for various numbers, and an example of how to write one number (5625) in that system. (Chinese is written up to down; the number is written in the vertical column on the right.)
   (a) From the example, figure out how the number system works, and write 367 and 6024 in that system. Is this a simple grouping system? A positional system? Neither? Explain.
   (b) Also write 367 and 6024 in the Egyptian and Babylonian systems.

2. I Kings 7:23 (and II Chronicles 4:2), given below, describes the building of a large circular reservoir for use in the temple. Use the information in this verse to determine the value of π used by the Hebrews. (Recall that π is the ratio of the circumference to the diameter of a circle.)

"He made the Sea of cast metal, circular in shape, measuring ten cubits from rim to rim and five cubits high. It took a line of thirty cubits to measure around it."

3. Plato's dialogue *Theaetetus* refers to the story of the discovery that $\sqrt{2}$ is irrational, and goes on to discuss a later mathematician Theodorus' contributions. Theodorus proved that the square roots of *all* the whole numbers up to 17 (except 1, 4, 9, 16) are irrational. Plato says next: "Then for some reason he stopped." Many have tried to explain why he stopped at 17. One fanciful (but unlikely) explanation is connected with the spiral diagram at the end of this assignment. We start with a right-angled triangle with side lengths equal to 1 unit (triangle ①). What is the length of the hypotenuse? Next, build another right-angled triangle by extending a line of 1 unit off the hypotenuse (triangle ②). What is the length of the new hypotenuse? Keep doing this; what is the length of each successive hypotenuse? Draw a *very accurate* diagram on paper and continue to 17 and beyond. What happens to the diagram you reach the triangle with hypotenuse equal to $\sqrt{17}$?

4. In the manner of Greek geometrical algebra, use a diagram to prove the following algebraic facts:
   (a) $(x+2y)^2 = x^2 + 4xy + 4y^2$
   (b) $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ (For this you'll need a 3-D picture!)

5. (a) One of the most famous non-Western "proofs" of the Pythagorean theorem comes from the Indian mathematician Bhaskara (AD 1150). It consists only of the diagram below left, and the single word "Behold!". You can see why this diagram proves the theorem by labelling the sides of the four identical triangles *a, b,* and *c* respectively. The area of the oblique square in the middle is obviously c^2. But you can also find its area by taking the area of the whole square and subtracting the four triangles from it. Do this, and show that the area you get for the leftover square is $a^2 + b^2$.

   (b) A similar Chinese "proof", from Liu Hui in the third century AD, uses the diagram below right. Ignore the rectangular grid in the diagram, and again label the triangle sides *a,b,* and *c*. Once again the oblique square's area is obviously c^2. Find its area another way by adding up the five regions *inside* of the oblique square. You should get $a^2 + b^2$.

Sample Problem Set Questions (diagrams not included)

Each of these questions is based on an article dealing with a particular theme. For Question 1 the article describes a proposal for how deductive mathematics might have arisen in Egypt; for Questions 2 and 3 the article (by me) discusses how religious beliefs have affected the way several pre-modern cultures did mathematics; and for Question 4, the article defined the term “the feminine mathtique” and explored the roles (or lack thereof) of women in mathematics through history.

1. There are two main schools of thought on the value of mathematics in one’s education: that it is of practical use, and that it is good training for the mind. Which of these would have been the dominant philosophy in (a) Egypt; (b) Babylon; and (c) Greece? Explain. Also, describe which philosophy best characterizes the approach underlying the mathematical education you had in elementary and high school.

2. (i) The article claims that some pre-Socratic beliefs may be called religious in nature. Which beliefs are they? (ii) The case depends strongly on what one means by “religious belief”. Do the definitions described in the article coincide with your previous notions of religion? Now that you’ve seen them, do they seem plausible to you? Elaborate.

3. Critique the paper’s main argument that religious belief affects mathematics through the influence of broad cultural, ultimately religious, assumptions. Are you convinced? Why/why not?

4. Do you believe that social attitudes toward women in mathematics is an issue which ought to be addressed through changes in social and/or educational policy? If so, what measures would be appropriate? If not, why not?

Sample Paper Topics

These are taken from a collection of topics I distribute to the class as a feeder for ideas. Generally I would hope that a student could find a topic on his/her own, but these are usually a good spark to get the neurons firing.

Mathematics, Astronomy, and the Great Pyramid. It is hard to imagine that the ancient Egyptians could have built the pyramids with as little mathematical sophistication as the existing texts show, although plausible reconstructions do exist. However, a renegade group of scholars has recently revived a long-standing conjecture that the pyramids in general, and the Great Pyramid at Gizeh in particular, reveal a great deal of knowledge of mathematics, and particularly astronomy. The book below, written by these scholars, presents their case (it has also been shown in a documentary on the Discovery Channel). Make sure to explore the matter more fully, including the view of the opponents, and past views on the issue.

Robert Bauval and Adrian Gilbert, *The Orion Mystery*, London: William Heinemann, 1994.

Archimedes' Quadrature of the Parabola; Islamic Connections; Referee a Paper. One of the most beautiful pieces of mathematics to come from Ancient Greece was Archimedes' determination of the area of a parabola. A few years ago I refereed a paper on this topic submitted to *Mathematics Magazine*. If you have some mathematical experience, you may appreciate the challenge of explaining Archimedes' work and its importance, as well as reviewing and criticizing the paper yourself. This topic is for those who would like to find out a bit what it's like to be a scholar. Be sure to see me first!

π. No number has brought more mystery and fascination throughout history than π. It has attracted the minds of the greatest scientists and the greatest cranks, and continues to turn up in mathematics where you least expect it. What other number has several web sites devoted to it? You may choose various topics here: for instance, the various calculation attempts (including the most recent attempts by the Borwein brothers, among others), its appearance in other mathematical topics, etc. If you're interested, these books will help you focus your interest and provide good source material:

J. L. Berggren, Jonathan Borwein, and Peter Borwein, eds. *Pi: A Source Book*, Berlin: Springer Verlag, 1997.

Petr Beckmann, *A History of Pi*, St. Martin's Press, 1976.

Eudoxus and Dedekind. The logical problems with the differences between numbers and geometric lengths go back to the difficulty encountered by the Pythagoreans over the irrationality of $\sqrt{2}$. Eudoxus cleared up the problem for the ancient Greeks, with full logical rigour. In the late 19th century, Richard Dedekind proposed a modern counterpart to this theory. Compare and contrast the two methods, and describe Dedekind's motivations for developing a new theory.

Start on Eudoxus with 5-5 of Howard Eves' book. Start on Dedekind in any text on the history of mathematics. Also consult Dedekind's *Essays on the Theory of Numbers*. I have a copy of this book. The mathematics for this topic is more difficult than average.

The Golden Ratio in Greek Mathematics. Many aspects of Greek art and architecture are based on the unusual number $(\sqrt{5}-1)/2$, known to the Greeks as "mean and extreme ratio" and now as the "golden ratio". Many interesting relations based on this number were discovered by the Greeks, and they used it often in art. Plato refers to it in his philosophical work *The Republic*. Examine the mathematical, artistic, and philosophical threads surrounding the golden ratio.

There are several sources on Greek and/or recreational mathematics containing information on this topic. See me if you have trouble finding material.

H. E. Huntley. *The Divine Proportion: A Study in Mathematical Beauty*. New York: Dover, 1970.

The Discovery of Non-Euclidean Geometry. In the nineteenth century, after more than two millenia of trying to prove Euclid's Parallel Postulate, mathematicians began to realize that it may in fact not be true, or at least that it was consistent to assume it was false. This led to worlds of new types of strange geometries, one of which is nowadays considered to be a better match to our physical space than the geometry we are used to. Explore some of the earlier attempts to prove the Parallel Postulate, outline the physical and philosophical importance of the possible existence of "other geometries", and describe society's reactions to this new type of mathematics.

Marvin Jay Greenberg. *Euclidean and Non-Euclidean Geometries: Development and History.* 3rd edition. New York: W. H. Freeman, 1993.

French Mathematics in the Eighteenth Century. France was the centre of mathematical learning in the 18th century, with famous names like Lagrange, Laplace, and d'Alembert. It was an exuberant period, characterized by the strengthening of the belief that mathematics could "decode" the natural world. At the same time, dramatic upheavals in political and philosophical movements were happening in France, culminating in the French Revolution. Place the French mathematical culture in its broader context, and give a picture of how they inter-relate.

The History of Cryptography. The art of keeping secrets goes back as far as human conflict; in other words, recorded history. Simple codes and ciphers may be found in ancient and medieval texts, primarily for the purpose of preserving military and state secrets. The theory of cryptography really took off in the 20th century; some credit the Allied success in World War II to a group of British scientists and mathematicians who broke the Germans' "Enigma" code and learned Nazi military secrets. Today, Internet commerce relies on more sophisticated encryption, and has made number theory and abstract algebra part of the backbone of everyday high-technology life.

Begin with (but go beyond) Simon Singh, *The Code Book*. New York: Doubleday, 1999.

Descartes' Dream and the Information Age. René Descartes' philosophical perspective set off an entire tradition of rational belief in the power of mathematics to explain creation. The belief took hold of society very quickly, but some feel that the age of computing technology has actually increased its hold. Trace the path of Descartes' dream through the 17th to 20th centuries and discuss its effects on our culture. You may also wish to discuss how some post-modernist beliefs could be considered to be a reaction against Descartes' rationalism.

P. Davis and R. Hersh, *Descartes' Dream*: *The World According to Mathematics*. Boston: Houghton Mifflin, 1988.

Edwin Abbott's *Flatland*. Late in the 19th century, Edwin Abbott, a theologian with an interest in mathematics, wrote a charming "fairy tale" about life in a two-dimensional plane, in order to extend our imagination to the possibility of an existence in four or more dimensions. The book had theological and societal messages beyond and inside the mathematical content, and became more relevant with 20th century physical theories. Read the book and the article below, and incestigate Abbott's motivations and context.

Flatland (and a 1965 sequel *Sphereland* by Dionis Burger) should be easy to find. Also read an article by Thomas Banchoff (available from me), which includes an extensive bibliography on Abbott, and another recent sequel by Ian Stewart.